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THERMAL WAVES IN A SUPERSONIC BOUNDARY LAYER WITH SELF-INDUCED PRESSURE*

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The linear problem of temperature perturbation on a flat plate in a supersonic stream of gas is considered using the asymptotic theory of plane flow in a boundary layer with free interaction. Problem is solved by applying the Fourier transform in the longitudinal coordinate. The effect of wall temperature variation on the physical characteristics of gas flow in the boundary layer is investigated. Numerical application of the inverse Fourier transform is used for determining the effect of a given temperature variation on pressure distribution.

1. Statement of the problem. Let us consider the supersonic flow of gas at Mach number higher than unity over a plate. We use the Cartesian system of coordinates with the x axis directed along the plate, the y axis normal to it, and the origin at the point of beginning of the perturbed temperature region on the plate. We denote time by t, the velocity vector components by u and v, density by ρ, pressure by p, temperature by T, and by λ the first viscosity coefficient. Indices ∞ and w denote parameters in the unperturbed stream and at the plate, respectively.

In the theory of free interaction the motion of gas in steady /1 – 3/ and unsteady /4 – 7/ state problems is usually considered in three characteristic regions, viz. the upper region where the viscosity and heat conduction effects are small and the flow is irrotational; the intermediate region of vortex flow but where the effect of dissipative factors can be neglected, and the lower region in direct contact with the body where the flow pattern essentially depends on viscosity. Solution of asymptotic equations for the lower region is the most difficult. In the first two regions the flow is quasi-steady and only parametrically dependent on time, while in the lower region it is essentially unstable and defined by equations of the boundary layer of a compressible gas whose pressure gradient is not a priori known, and has to be determined in the course of the problem solution under conditions of interaction with the external flow.

In the investigations of unsteady flows in the region of interaction in /4 – 7/ it was assumed that the body surface was thermally insulated and that in the region free of interaction large temperature gradients were absent. On these assumptions it is possible to consider density and viscosity as constant, and in the layer next to the wall to be determined by the solution in the intermediate region. Allowance for gas compressibility in the region of free interaction enables us to widen the class of this type of problems that involve wall temperature variation. The steady state problem of temperature discontinuity at a plate subjected to a supersonic flow of viscous gas was considered in /8/ using the theory of free interaction. Patterns of pressure distribution induced by abrupt temperature changes of the plate were also determined there.

Below, we investigate the linear problem of perturbation propagation in an unsteady boundary layer, induced by temperature perturbation on the plate surface.

Let us assume that the (gas) specific heat at constant pressure is constant, the Prandtl number is unity, and that the coefficients of viscosity λ and thermal conductivity k^T linearly depend on temperature in conformity with Chapman's law

$$\lambda/\lambda_\infty = cT/T_\infty,\ k^T/k_\infty{}^T = cT/T_\infty,\ c = \text{const}$$

Assuming further the gas to be perfect, we can eliminate the temperature from the system of equations that define the flow next to the plate and formulate the boundary value problem for perturbed density.

On the above assumptions we can formulate the equations that define the plane unsteady motion of compressible gas in a boundary layer with self-induced pressure, together with boundary conditions, in terms of transformed dimensionless variables, in the form /9/

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$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}=0 \tag{1.1}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{\partial u}{\partial y}\right),\ \frac{\partial p}{\partial y}=0$$

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}=\rho\frac{\partial}{\partial y}\left(\frac{1}{\rho^2}\frac{\partial\rho}{\partial y}\right)$$

$$y\to\infty: u\to y-\int\limits_{-\infty}^{x}p\,dx,\quad \rho\to 1 \tag{1.2}$$

$$x\to-\infty: u\to y,\ p\to 0,\ \rho\to 1$$

$$y=0: u=0,\ v=0,\ \rho=\rho_w(x,t)$$

2. Free oscillations. Let us first consider the simpler boundary value problem in which the plate temperature is constant, i.e. by setting $\rho_w=1$ in the last of conditions (1.2). We shall seek wave solutions periodic in x of the form

$$u=y-ae^{i(\Omega t+kx)}G(y),\ v=ake^{i(\Omega t+kx)}F(y) \tag{2.1}$$

$$p=ae^{i(\Omega t+kx)},\ \rho=1+ae^{i(\Omega t+k t x)}\Phi(y)$$

assuming the perturbation amplitude a to be small. The substitution of expressions (2.1) into Eqs.(1.1) and boundary conditions (1.2) together with the modified condition for density at the plate, yields the following system of equations and boundary conditions:

$$(\Omega+ky)\Phi-kG+k\frac{dF}{dy}=0 \tag{2.2}$$

$$d^2G/dy^2-i(\Omega+\kappa y)+ikF+ik=-d\Phi/dy\quad d^2\Phi/dy^2-i(\Omega+ky)\Phi=0$$

$$y=0: G(0)=0,\ F(0)=0,\ \Phi(0)=0$$

$$y\to\infty: \Phi(y)\to 0,\ G(y)\to 1/(ik)$$

Furthermore, considering that perturbations must dampen as $x\to-\infty$, it is necessary to assume that $\operatorname{Im}k<0$. The last equation for Φ in system (2.2) can be separated, and after the substitution of variable $z=i^{1/3}\Omega/k^{2/3}+(ik)^{1/3}y$ reduces to the Airy equation. Its solution, which satisfies the boundedness condition as $y\to\infty$ can be straightaway expressed in terms of Airy's function

$$\Phi(z)=c_0\operatorname{Ai}(z)$$

To satisfy the condition for $\Phi(z)$ when $y=0$ it is necessary to satisfy the condition

$$\operatorname{Ai}(i^{1/3}\Omega/k^{2/3})=0 \tag{2.3}$$

The case of $c_0=0$ corresponds to the problem of propagation of purely mechanical oscillations, which was considered earlier in /4–6/. If $c_0\neq 0$, constants Ω and k must satisfy condition (2.3), which implies the relation $i^{1/3}\Omega/k^{2/3}=z_n$, where z_n are roots of Airy's function that lie on the negative part of the real axis. Hence condition (2.3) is a dispersion relation that determines the eigenvalues of the considered here boundary value problem. Slitting plane k along the positive part of the imaginary axis will show that by virtue of the assumption that $\operatorname{Im}k<0$ and condition (2.3) $\pi/6<\arg\Omega<5\pi/6$, i.e. $\operatorname{Im}\Omega>0$ and the oscillation amplitude dampens with time. If we introduce the phase velocity $c=-\Omega/k$, then $-5\pi/6<\arg c<-7\pi/6$, i.e. $\operatorname{Re}c>0$ which corresponds to the downstream propagation of waves.

Substituting the solution for Φ into the first two equations of system (2.2) and solving the obtained system with allowance for the remaining boundary conditions, for the longitudinal velocity we obtain

$$G(z)=-\left[c_0+\frac{(ik)^{2/3}}{\operatorname{Ai}'(\zeta)}\right](ik)^{-4/3}[I(\zeta)-I(z)] \tag{2.4}$$

$$\zeta=\frac{i^{1/3}\Omega}{k^{2/3}},\quad I(\zeta)=\int\limits_{\zeta}^{\infty}\operatorname{Ai}(z)\,dz$$

$$c_0=-\frac{(ik)^{4/3}I(\zeta)+\operatorname{Ai}'(\zeta)}{(ik)^{2/3}\operatorname{Ai}'(\zeta)I(\zeta)} \tag{2.5}$$

The numerator in formula (2.5) is a dispersion relation which was obtained in the investigation of perturbation propagation in the constant density boundary layer next to the plate

/4,6/. If Ω and k satisfy that dispersion relation, then $c_0 = 0$ and, as expected, formula (2.4) come to the solution derived in /6/.

3. Solution of the linear problem with varying plate temperature. Let us assume that the plate temperature variation is periodic in time, so that density variation next to the plate is defined by the formula

$$\rho_w(x, t) = 1 + ae^{i\Omega t} f(x)$$

where Ω is the dimensionless oscillation frequency, a is a small parameter and function $f(x)$ defines the oscillation form. By virtue of the assumption of smallness of parameter a the problem can be linearized, representing the solution in the form of expansions in

$$u = y + au' + \ldots, \quad v = av' + \ldots, \quad \rho = 1 + a\rho' + \ldots, \quad p = ap' + \ldots$$

The substitution of these expansion into Eqs.(1.1) and boundary conditions (1.2) yields for the first approximation functions the following system of equations and boundary conditions:

$$\frac{\partial\rho'}{\partial t} + y\frac{\partial\rho'}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \qquad \frac{\partial p'}{\partial y} = 0$$

$$\frac{\partial u'}{\partial t} + y\frac{\partial u'}{\partial x} + v' = -\frac{\partial p'}{\partial x} - \frac{\partial\rho'}{\partial y} + \frac{\partial^2 u'}{\partial y^2}$$

$$\frac{\partial\rho'}{\partial t} + y\frac{\partial\rho'}{\partial x} = \frac{\partial^2\rho'}{\partial y^2}$$

$$x \to -\infty: \{u', v', \rho', p'\} \to 0$$

$$y \to \infty: u' \to -\int_{-\infty}^{x} p'(x, t)\,dx, \qquad \rho' \to 0$$

$$y = 0: u' = 0, \; v' = 0, \; \rho' = e^{i\Omega t} f(x)$$

In this system the equation for ρ' can be separated and integrated independently of the remaining, and simple transformations enable us to eliminate ρ', v' and p', and obtain for u' the equation

$$\frac{\partial^2 u'}{\partial t\partial y} + y\frac{\partial^2 u'}{\partial x\partial y} = \frac{\partial^3 u'}{\partial y^3} \tag{3.1}$$

whose solution will be sought in the form

$$\rho' = e^{i\Omega t} R(x, y), \; u' = e^{i\Omega t} U(x, y), \; p' = e^{i\Omega t} P(x) \tag{3.2}$$

For the determination of function R we use the Fourier transform

$$R^*(k, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} R(x, y)\,dx$$

and similar transforms for U and P. This actually presumes that perturbations of ρ', u' and p' dampen as $x \to \infty$. As the result, we obtain for $R^*(k, y)$ an equation which, as previous, reduces to the Airy equation after the introduction of the variable z. We represent its solution, with allowance for the boundary conditions for density, in the form

$$R^*(k, z) = \frac{f^*(k)}{\mathrm{Ai}(\zeta)}\mathrm{Ai}(z), \qquad \zeta = \frac{i^{1/3}\Omega}{k^{2/3}}$$

where $f^*(k)$ the Fourier transform of function $f(x)$, which defines the form of density oscillations at the boundary. The inverse transformation yields

$$R(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{f^*(k)\,\mathrm{Ai}(z)\,e^{ikx}}{\mathrm{Ai}'(\zeta)}\,dk \tag{3.3}$$

Dealing similarly with Eq.(3.1) for velocity, and taking into account solution (3.3), we obtain for $U^*(k, z)$ the formula

$$U^*(k, z) = \frac{(ik)^{1/3} P^*(k)\,\mathrm{Ai}(\zeta) + (ik)^{-1/3} f^*(k)\,\mathrm{Ai}'(\zeta)}{\mathrm{Ai}(\zeta)\,\mathrm{Ai}'(\zeta)}\,[I(\zeta) - I(z)] \tag{3.4}$$

Using the boundary condition for velocity as $y \to \infty$, we obtain from formula (3.4) the Fourier transform for pressure $P^*(k)$ and, consequently, also

$$P(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} k^2 f^*(k)\, \Phi(\Omega, k)\, e^{ikx}\, dk \tag{3.5}$$
$$\Phi(\Omega, k) = \frac{I(\zeta)\,\mathrm{Ai}'(\zeta)}{(ik)^{4/3}[\mathrm{Ai}'(\zeta) + (ik)^{4/3} I(\zeta)]\,\mathrm{Ai}(\zeta)}$$

For calculating integral (3.5) in the complex plane it is necessary to know the poles of integrands which are determined by the roots of the denominator in $\Phi(\Omega, k)$. Roots of the expression in brackets correspond to eigenvalues in the problem of free mechanical oscillations /6/, while the roots of $\mathrm{Ai}(\zeta) = 0$ yield eigenvalues in the considered above problem of free oscillations in a compressible boundary layer with self-induced pressure. The perturbed unsteady pressure according to formula (3.2) is determined as follows:

$$p'(x, t) = \cos \Omega t \,\mathrm{Re}\,[P(x)] - \sin \Omega t \,\mathrm{Im}\,[P(x)] \tag{3.6}$$

4. The determination of pressure. For further analysis it is advantageous to define function $f(x)$ in specific form. We specify it in the form of a triangle with parameters a and b

$$f(x) = \begin{cases} 0, & x \leqslant 0 \\ 2x, & 0 \leqslant x \leqslant b \\ 2b\,(a - x)/(a - b), & b \leqslant x \leqslant a \\ 0, & x \geqslant a \end{cases}$$

In this case

$$P(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \left[1 - \frac{a}{a-b} e^{-ikb} + \frac{b}{a-b} e^{-ika}\right] \Phi(\Omega, k)\, e^{ikx}\, dk \tag{4.1}$$

hence the determination of pressure reduces to the calculation in the complex plane k of integrals of the form

$$J(x) = \int_{-\infty}^{+\infty} \Phi(\Omega, k)\, e^{ikx}\, dk \tag{4.2}$$

For separating the single-valued branch of the integrand we slit plane k along the positive part of the imaginary axis /7/. One root of the denominator of the integrand in (4.2) is obviously $k = 0$, and the roots of Airy's function $\mathrm{Ai}(\zeta)$ are known and lie on the negative part of the real axis in the ζ plane and in the k plane, along the ray issuing from the coordinate origin at angle $5\pi/4$. The roots of expression in brackets in $\Phi(\Omega, k)$ were analyzed in detail in /7/ in connection with the problem of a vibrator in a supersonic boundary layer. That expression has one root in the fourth quadrant and a denumerable number of roots in the second.

We select the integration path C in the lower half-plane (Fig.1) on the basis of properties of roots of the integrand of integral (4.2). This path consists of the real axis segment between $-R$ and R and of an arc of circle of radius R, by-passing the coordinate origin along semicircle C_ε of radius ε. Applying the Cauchy theorem on residues and increasing radius R of the larger semicircle to infinity while decreasing radius ε of the small semicircle to zero, we obtain for $x < 0$

$$\int_{-\infty}^{+\infty} \Phi(\Omega, k)\, e^{ikx}\, dk = B - 2\pi i K(\Omega, k_1^*)\, e^{ik_1^* x} \tag{4.3}$$
$$K(\Omega, k_1^*) = \frac{3}{2} \frac{I(\zeta_1^*)\,\mathrm{Ai}'(\zeta_1^*)/\mathrm{Ai}(\zeta_1^*)}{(ik)^{5/3}[\zeta_1^*(i - \Omega/k_1^{*2})\,\mathrm{Ai}(\zeta_1^*) + 2iI(\zeta_1^*)]}$$

where ζ_1^* is the pole of integrand in the ζ plane, which corresponds to k_1^*, and $K(\Omega, k_1^*)\, e^{ik_1^* x}$ is the residue of the integrand at point k_1^*. The constant B is the value of the integral over C_ε as the radius ε of circle approaches zero. The value of B is unimportant, since it cancels out with the substitution into (4.1) of all three terms appearing in it.

For calculating integral (4.2) for $x > 0$ we apply the method recommended in /7/ for similar integrals. Application of the theorem on residues to path C for $x > 0$ yields

$$J(x) = B - 2\pi i K(\Omega, k_1^*)\, e^{ik_1^* x} + J_C(\Omega, x) \tag{4.4}$$
$$J_C(\Omega, x) = \lim_{R \to \infty} \int_{C_R} \Phi(\Omega, k)\, e^{ikx}\, dk$$

In the case of large k and finite $\Omega \geqslant 1$ the integrand can be represented in the form of series in ζ

$$\Phi(\Omega, k) = -\frac{\zeta^4}{\Omega^2} \sum_{N=0}^{\infty} c_N \zeta^N \tag{4.5}$$

with coefficients c_N, which depend on Ω, are determined in terms of known coefficients of expansions $\mathrm{Ai}(\zeta)$, $\mathrm{Ai}'(\zeta)$ and $I(\zeta)$. Integration by part of the series, we obtain the formula

$$J_C(\Omega, x) = -\sqrt{3}\,\Gamma(1/3)\,\Omega^2 x^{5/3} \sum_{m=0}^{\infty} \frac{(i\Omega)^{3m} x^{2m} c_{3m}}{(2m+5/3)(2m+2/3)+\ldots 2/3} + \tag{4.6}$$

$$2\pi i \Omega x \sum_{m=1}^{\infty} \frac{(i\Omega)^{3m} x^{2m} c_{3m-1}}{(2m+1)!} -$$

$$i\sqrt{3}\,\Gamma(2/3)\,\Omega^3 x^{7/3} \sum_{m=0}^{\infty} \frac{(i\Omega)^{3m} x^{2m} c_{3m+1}}{(2m+7/3)(2m+4/3)+\ldots 1/3}$$

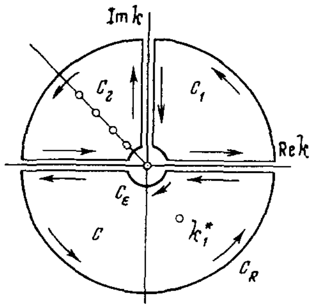

Fig.1

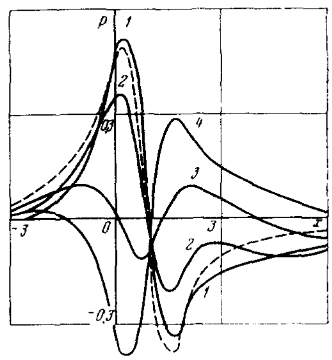

Fig.2

This series is rapidly convergent and convenient for computer calculations for $\Omega \geqslant 1$. When $\Omega < 1$, it is necessary to alter expansion (4.5) by introducing the variable $\zeta_1 = \zeta/\Omega$, which reduces it to the form

$$\Phi(\Omega, k) = -\zeta_1^4 \sum_{N=0}^{\infty} c_N' \zeta_1^N \tag{4.7}$$

where coefficients c_N' are formally linked to c_N by the relation $c_N' = c_N \Omega^{2+N}$. The series for calculating $J_C(\Omega, x)$ can be obtained form (4.6) by the formal substitution in it of c_N' for c_N. As in (4.7), the dependence on Ω appears in this series only in terms of coefficients c_N'.

Finally, the substitution of results of calculations by formulas (4.3), (4.4) and (4.6) into formula (4.1) yields

$$R(x) = \frac{1}{\pi}\left\{-2\pi i K(\Omega, k_1^*)\, e^{ik_1^* x}\left[1 - \frac{a}{a-b} e^{-ik_1^* b} + \frac{b}{a-b} e^{-ik_1^* a}\right] + \right. \tag{4.8}$$

$$\left. \theta(x) J_C(\Omega, x) - \frac{a\theta(x-b)}{a-b} J_C(\Omega, x-b) + \frac{b\theta(x-a)}{a-b} J_C(\Omega, x-a)\right\}$$

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

Pressures calculated by formulas (3.6) and (4.8) for $a = 2$, $b = 1$, $\Omega = 1$ and various instants of the dimensionless time $t = 0, \pi/4, \pi/2, 3\pi/4$ are shown in Fig.2 by curves $1-4$, respectively.

5. Determination of pressure in the steady state case. Formula (4.8) shows that the derived solution parametrically depends on Ω. Setting in the input formulas (4.1) and (4.2) $\Omega = 0$ $(\zeta \to 0)$, we obtain for the steady state case

$$P(x) = -\frac{\Gamma(2/3)}{\pi 3^{2/3}\Gamma(4/3)}\left\{J_0(x) - \frac{a}{a-b} J_0(x-b) + \frac{b}{a-b} J_0(x-a)\right\} \tag{5.1}$$

$$J_0(x) = \int_{-\infty}^{+\infty} \frac{e^{ikx}\, dk}{(ik)^{4/3}\left[(ik)^{4/3} - b_0\right]}, \quad b_0 = \frac{1}{3^{1/3}\Gamma(4/3)}$$

where roots of the integrand denominator are $k_0 = 0$, $k_1 = -ib_0^{3/4} = -i\,[3^{1/3}\Gamma(4/3)]^{-3/4}$. For calculating $J_0(x)$ we select in the lower half-plane the same integration path C for $x < 0$. When $x > 0$ the path consists of two branches C_1 and C_2 that lie in the upper half-plane (Fig.1). They comprise segments of the real axis, of the arc of circle of the large radius R, the slit edge, and the arc of circle of the small radius ε. Note that as $\varepsilon \to 0$, we have at the by-pass of point $k = 0$ in (5.1) terms with the singularity $O(\varepsilon^{-1/3})$ in each integrals $J_0(x)$ in (5.1). However the substitution of all three derived integrals into (5.1) cancels these singularities, and we obtain for $P(x)$ an expression that contains only residues at point k_1 and integrals along edges of the slit. We can finally write

$$P(x) = \frac{3^{2/3}\Gamma(2/3)}{2b_0^{1/4}}\Big\{[1-\theta(x)]\exp(b_0^{3/4}x) - \frac{a}{a-b}[1-\theta(x-b)]\times \tag{5.2}$$

$$\exp(b_0^{3/4}(x-b)) + \frac{b}{a-b}[1-\theta(x-a)]\exp(b_0^{3/4}(x-a)) -$$

$$\frac{2b_0^{1/4}}{3\pi}\Big[\theta(x)J_R(x) - \frac{a}{a-b}\theta(x-b)J_R(x-b) +$$

$$\frac{b}{a-b}\theta(x-a)J_R(x-a)\Big]\Big\}$$

$$J_R(x) = 3\sqrt{3}\Gamma(2/3)\,x^{1/3} + \frac{\sqrt{3}}{b_0^{1/4}}\int_0^\infty \frac{t^{4/3}\exp(-b_0^{3/4}xt)}{1+t^{4/3}+t^{8/3}}\,dt$$

Pressure distribution in the steady state case calculated by formula (5.2) with $a = 2$ and $b = 1$ is shown in Fig.2 by the dash line.

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